GCSE Maths – Algebra

Factorising Linear and Quadratic Expressions

Notes

WORKSHEET

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Factorising Linear and Quadratic Expressions

Factorising

Factorising is the **opposite** operation to expanding brackets so a factorised expression will always contain a set of brackets. There are some simple steps to follow when factorising an expression:

- Take out the **highest common factor** of the terms in the expression and place them outside the brackets.
- For **each letter**, take the **highest power** (e.g. x, x^2) that is present in **every** term. Write this outside the brackets and then open the brackets and fill in everything that is needed to **reproduce** each **original term**.

As factorising is the opposite of expanding brackets, we can **check** we have done it right by **multiplying out the brackets** and it should leave us with the **original expression**.

Example: Factorise the expression $4x^2 + 8x$

1. Work out the **highest common number factor** for the terms in the expression. Factorise the HCF outside the brackets by dividing it out of each term.

   The highest common factor that goes into both terms is 4 *so*

$$4x^2 + 8x = 4(x^2 + 2x)$$

2. Work out the **highest power** of x that will go into both terms. Factorise this power of x outside the brackets by dividing it out of each term.

   The highest power of x *that will go into both terms is* x. *We factorise this out of the brackets:*

$$4x^2 + 8x = 4(x^2 + 2x) = 4x(x + 2)$$

   The final expanded answer is $\mathbf{4x(x+2)}$.

3. **Check** this factorisation by multiplying out the brackets.

   By multiplying out the brackets, if we return to our original expression of $4x^2 + 8x$ *then we know we have factorised the expression correctly.*

$$\mathbf{4x(x+2)} = (4x \times x) + (4x \times 2) = \mathbf{4x^2 + 8x}$$

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Example: Factorise the expression $6x^2y + 3xy^2$

1. Work out the **highest common number factor** for the terms in the expression. Factorise the HCF outside the brackets by dividing it out of each term.

   The highest common factor that goes into both terms is 3 *so*

$$6x^2y + 3xy^2 = 3(2x^2y + xy^2)$$

2. Work out the **highest power** of x and y that will go into both terms. Factorise this power of x and y outside the brackets by dividing it out of each term.

   The highest power of xy *that will go into both terms, is* xy.

$$6x^2y + 3xy^2 = 3(2x^2y + xy^2) = 3xy(2x + y)$$

   The final expanded answer is $\mathbf{3xy(2x + y)}$.

4. **Check** this factorisation by multiplying out the brackets**.**

   By multiplying out the brackets, if we return to our original expression then we know we have factorised the expression correctly.

$$\mathbf{3xy(2x + y)} = (3xy \times 2x) + (3xy \times y) = \mathbf{6x^2y + 3xy^2}$$

Factorising using Difference of Two Squares

When we have an expression with one **squared term subtracted** from **another squared term**, we can use the difference of two squares (DOTS) to help factorise. The DOTS **general formula** is:

$$\mathbf{a^2 - b^2 = (a + b)(a - b)}$$

In this formula the letters a and b just represent any algebraic terms. In problems, these are usually replaced with various different numbers and letters.

Example: Factorise the expression $x^2 - 1$

1. Check if you can factorise the expression normally. Check if there are **any common factors** that can be taken out.

   For $x^2 - 1$ *there are no common factors which can be taken out.*

2. Check if the expression is one squared term subtracted from another squared term. If so, factorise using the **difference of two squares** general formula $\mathbf{a^2 - b^2 = (a + b)(a - b)}$**.**

$$x^2 - 1 = x^2 - 1^2$$

   Using the general formula with $a = x$, $b = 1$, we get

$$x^2 - 1 = x^2 - 1^2 = \mathbf{(x + 1)\,(x - 1)}$$

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Example: Factorise the expression $16a^2 - 25b^2$

1. Check if you can factorise the expression normally. Check if there are **any common factors** that can be taken out.

   For $16a^2 - 25b^2$ *there are no common factors which can be taken out.*

2. Check if the expression is one squared term subtracted from another squared term. If so, factorise using the **difference of two squares** general formula $\boldsymbol{a^2 - b^2 = (a+b)(a-b)}$**.**

   For the expression $16a^2 - 25b^2$*, we have*

$$16a^2 - 25b^2 = (4a)^2 - (5b)^2.$$

   Using the DOTS general formula with $a = 4a$*,* $b = 5b$*, we get*

$$16a^2 - 25b^2 = \mathbf{(4}\boldsymbol{a}\mathbf{+5}\boldsymbol{b}\mathbf{)\,(4}\boldsymbol{a}\mathbf{-5}\boldsymbol{b}\mathbf{)}$$

Example: Factorise the expression $3y^2 - 75$

1. Check if you can factorise the expression normally. Check if there are **any common factors** that can be taken out.

   For $3x^2 - 75$ *there are common factors which can be taken out.*
   The common factor 3 *can be taken out, giving:*

$$3y^2 - 75 = 3(y^2 - 25)$$

2. Check if the expression is one squared term subtracted from another squared term. If so, factorise using the **difference of two squares** general formula $\boldsymbol{a^2 - b^2 = (a+b)(a-b)}$**.**

   For the expression $3(y^2 - 25)$*, we have*

$$3(y^2 - 25) \;=\; 3\,[(y)^2 - (5)^2].$$

   Using the DOTS general formula with $a = y$, $b = 5$, we get

$$3(y^2 - 25) = \;3\,[(y)^2 - (5)^2] = 3[(y+5)\,(y-5)] = \mathbf{3(}\boldsymbol{y}\mathbf{+5)(}\boldsymbol{y}\mathbf{-5)}$$

Factorising Quadratics of the Form $x^2 + bx + c$

A quadratic is an expression where the highest power of x is 2. Factorising a quadratic expression involves writing it as the **product of two brackets**.

The general form of a quadratic is $\boldsymbol{ax^2 + bx + c}$, with each letter representing a number. We will be first looking at how to factorise this equation when $a = 1$.

There are a few general steps to remember when factorising:

1. If necessary, rearrange the expression so it is in the **standard form** of $\boldsymbol{ax^2 + bx + c}$
2. Equate the expression to a **pair of open brackets**: $(x\quad)\ (x\quad)$
3. Identify the **factors** that **multiply** to give the **'c' value** (the constant term) but also **add** to give the **'b' value** (the coefficient of x).
4. Write these values, say **p** and **q**, in the spaces in the brackets: $(x + p)\ (x + q)$
5. **Check** it is correct by **expanding** out the brackets. If correct, you will obtain the original quadratic expression.

Example: Factorise the expression $x^2 + 6x + 8$

1. If necessary, rearrange the expression into the **standard form**.

   This is not necessary here as it is already in the right form.
   The standard form is $\boldsymbol{ax^2 + bx + c}$. In this case, $a = 1$, $b = 6$ and $c = 8$.

2. Equate the expression to a pair of open brackets $(x\quad)(x\quad)$.

$$x^2 + 6x + 8 = (x\quad)(x\quad)$$

3. Identify the factors that multiply to give the 'c' value (the constant term) but also add to give the 'b' value (the coefficient of x).

$$x^2 + \mathbf{6}x + \mathbf{8}$$

$$1 \times 8 = 8 \text{ and } 1 + 8 = 7$$
$$-1 \times -8 = 8 \text{ and } -1 + -8 = -9$$
$$-2 \times -4 = 8 \text{ and } -2 + -4 = -6$$
$$2 \times 4 = \mathbf{8} \text{ and } 2 + 4 = \mathbf{6}$$

   The pair of numbers 2 and 4 give the product of $c = 8$ and sum of $b = 6$.

4. Write these values in the spaces in the brackets:

$$x^2 + 6x + 8 = (x + \mathbf{2})(x + \mathbf{4})$$

5. Check it is correct by expanding out the brackets. If correct, you will obtain the original quadratic expression.

$$\boldsymbol{(x+2)(x+4)} = x^2 + 4x + 2x + 8 = \boldsymbol{x^2 + 6x + 8}$$

   It is indeed correct, so the final answer is $\boldsymbol{x^2 + 6x + 8 = (x+2)\ (x+4)}$.

Factorising Quadratics of the Form $ax^2 + bx + c$, where $a \neq 1$ (Higher Only)

Factorising quadratics when the leading coefficient $a \neq 1$ is a little trickier than when $a = 1$. However, there are some general steps to follow which will help you get it right every time:

1. If necessary, rearrange the expression so it is in the **standard form** of $\boldsymbol{ax^2 + bx + c}$.

2. **Multiply** the value of a and c **together** so you get a value ac **.**

3. For this ac **value** you have obtained, write all the **factors** that **multiply to give this** $'ac'$ **value** but also **add/subtract** to give the **'b' value** (the coefficient of x).

4. Rewrite the **original expression** by **replacing the bx term** with **two** new **x terms** which each have one of these two values as their **coefficient**.

5. **Split** the expression into two by **grouping together pairs of terms** and then **factorise**. If done correctly, the two brackets for each pair should be **identical.**

6. Write the expression as the **product of two brackets**. The first bracket will be equal to the **common bracket** in both factorisations. The second bracket will be made up of the **coefficient terms** of each of these identical brackets. This will be clarified in the example.

7. **Check** by expanding the brackets. If the expression has been factorised correctly then on expansion you should obtain the original expression that you started with.

Example: Factorise the expression $4x^2 + 8x + 3$

1. If necessary, rearrange the expression so it is in the standard form of $ax^2 + bx + c$.

   It is already in the standard form:

   $$ax^2 + bx + c = \mathbf{4x^2 + 8x + 3}$$

   Here, $a = 4,\ b = 8$ *and* $c = 3$.

2. Multiply $\boldsymbol{a}$ with $\boldsymbol{c}$ and call this value $\boldsymbol{ac}$.

   $$ac = a \times c = 4 \times 3 = \mathbf{12}$$

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3. Write the **factors** of $\boldsymbol{ac}$ that will also **multiply** to give the $\boldsymbol{b}$ **value** of $b = 8$.

$$1 \times 12 = 12 \text{ and } 1 + 12 = 13$$
$$-1 \times -12 = 12 \text{ and } -1 + -12 = -13$$
$$-2 \times -6 = 12 \text{ and } -2 + -6 = -8$$
$$2 \times 6 = 12 \text{ and } 2 + 6 = 8$$
$$3 \times 4 = 12 \text{ and } 3 + 4 = 7$$
$$-3 \times -4 = 12 \text{ and } -3 + -4 = -7$$

The factors 2 *and* 6 *will add to give the value of* $b = 8$ *and multiply to give* $ac = 12$.

4. Write the values found as the coefficients of separate x terms.

We will write 2 *and* 6 *as* $2x$ *and* $6x$.

5. Write the original expression but with these x values as shown above replacing (splitting up) the bx term.

$$4x^2 + 8x + 3 = \mathbf{4x^2 + 2x + 6x + 3}$$

6. We will **split** the above into two different expressions by splitting it in half.

$$\mathbf{4x^2 + 2x + 6x + 3 = 4x^2 + 2x + 6x + 3}$$

7. Factorise each half of the expression separately:

$$\mathbf{4x^2 + 2x} \text{ **factorises** to } 2x(2x + 1)$$
$$\mathbf{6x + 3} \text{ **factorises** to } 3(2x + 1)$$

So,

$$\mathbf{4x^2 + 2x + 6x + 3 = 4x^2 + 2x + 6x + 3 = 2x(2x + 1) + 3(2x + 1)}$$

Each bracket formed above is identical which shows we have done it right.

8. Write the expression as the product of two brackets. The first bracket will be equal to the common bracket in both factorisations. The second bracket will be made up of the coefficient terms of each of these identical brackets.

$$\mathbf{2x(2x + 1) + 3(2x + 1) = 2x(2x + 1) + 3(2x + 1) = (2x + 3)(2x + 1)}$$

9. Check by expanding the brackets. If done correctly, you will obtain the original expression.

$$\mathbf{(2x + 3)(2x + 1) = 4x^2 + 2x + 6x + 3 = 4x^2 + 8x + 3}$$

So, the final answer is

$$\mathbf{4x^2 + 2x + 6x + 3 = (2x + 3)(2x + 1).}$$

Factorising Linear and Quadratic Expressions – Practice Questions

1. Factorise the following expressions:

   a) $5 - 10z$
   b) $-6e^3 - 18e^2$
   c) $6ef + 18efg - 3efg^2$
   d) $-48ab^2c^3 + 12b^2c^3 - 18a^2b^3c^2$

2. Factorise the following expressions:

   a) $z^2 - 1$
   b) $4p^2 - 25q^2$
   c) $16s^2 - 49t^2$
   d) $5x^2 - 125y^2$

3. Factorise the following expressions:

   a) $x^2 + 4x + 4$
   b) $x^2 + 3x + 2$
   c) $x^2 - 10x + 21$
   d) $-5x + x^2 - 24$

4. Factorise the following expressions:

   a) $2x^2 + 3x + 1$
   b) $7x^2 - 39x + 20$
   c) $18x^2 + 42x + 12$
   d) $40x^2 + 38x - 12$

Worked solutions for the practice questions can be found amongst the worked solutions for the corresponding worksheet file.